Gaussian Approach to (1+1) Dimensional φ^6 Solitons at Finite Temperature

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Using the Gaussian effective potential approach, φ^6 soliton solutions at finite temperature are studied for both the general case and the particular case $\lambda^2 = 2\xi m^2$. A critical temperature is found at which soliton solutions cease to exist. The effective potential together with the mass-gap equation are studied in detail, and comparison with existing work on this subject is made.

1. Introduction

Solitons at finite temperature have been studied by several authors [1–5]. Su et al. studied soliton solutions for φ^4 [1] and φ^6 [2] fields at finite temperature using the method of coherent state and the real time green function formalism. Kuczynski and Manka also studied the soliton solution for $\varphi^3 + \varphi^4$ fields using the Bogoliubov inequality [4]. In this paper we report the results of our study of the (1+1) dimensional real φ^6 solitons at finite temperature using the Gaussian effective potential (GEP) method. The motivation for this work is that the φ^6 field is of considerable interest because it may have three vacua (two real and a false one). Our work can be considered as an extension to our previous work on φ^4 solitons [5]. Roditi [6] has discussed the φ^6 field in the context of GEP but did not in particular discuss the φ^6 solitons. Also the mass gap equation was not at all considered by Roditi. Our normalization is also different from that of [6]. As far as the work of Su et al. [2, 3] is concerned, the effective potential in their work is quite different from that studied by us as they have not considered the contribution of the kinetic part. Also, in our opinion their normalization for the effective mass is not rigorous. This will be discussed later on. Apart from its simplicity, the GEP method has several advantages. It contains the one loop and the $1/N$ results as limiting cases.

2. Finite Temperature Effective Potential in (1+1) Dimension when $V(\varphi) = \eta\varphi^4 + \xi\varphi^6$

Let us now consider the Lagrangian density of (1+1) dimensional φ^6 fields

$$L = \tfrac{1}{2}\,\partial_\mu\,\partial^\mu\varphi - \tfrac{1}{2}\,m_B^2\,\varphi^2 - \eta\,\varphi^4 - \xi\,\varphi^6, \qquad (1)$$

where m_B is the bare mass and η (<0) and ξ are bare coupling constants. The corresponding Hamiltonian is given by

$$H = \int dx \left[\frac{1}{2}\,\dot\varphi^2 + \frac{1}{2}\,(\nabla\varphi)^2 + \frac{m_B^2}{2}\,\varphi^2 - \lambda_B\,\varphi^4 + \xi\,\varphi^6 \right]. \qquad (2)$$

Here $\lambda_B = -\eta > 0$.

The Gaussian effective potential (at finite temperature) is defined by

$$V_G^B(\varphi_0) = \min \frac{\mathrm{Tr}\,[\exp(-\beta H_\Omega)\,\mathscr{H}]}{\mathrm{Tr}\,[\exp(-\beta H_\Omega)]}, \qquad (3)$$

where

$$H_\Omega = \int dx \left[\frac{1}{2}\,\dot\varphi^2 + \frac{1}{2}\,(\nabla\varphi)^2 + \frac{\Omega^2}{2}\,\varphi^2 \right] \qquad (4)$$

and $\mathscr{H}$ is the Hamiltonian density corresponding to the Hamiltonian (2). In general, the classical soliton solution corresponding to (2) is given in terms of inverse elliptic functions, but for the particular case $\lambda^2 = 2\xi m^2$ the soliton solution is simple and is given by

$$\varphi(x) = \pm \frac{m}{\sqrt{2\lambda}}\,(1 \pm \tanh m x)^{1/2}. \qquad (5)$$

To use the Gaussian variational method we introduce the ansatz (in $(v+1)$ dimension)

$$\varphi(x) = \varphi_0 + \int \frac{\mathrm{d}^v k}{(2\pi)^v \, w_k(\Omega^2)} [a_\Omega(k) \, e^{-ikx} + a_k^+(k) \, e^{ikx}],$$ (6)

where

$$w_k(\Omega^2) = \sqrt{\Omega^2 + k^2}.$$ (7)

Both φ_0 and Ω^2 are variational parameters. a_Ω and a_Ω^+ are annihilation and creation operators, respectively, satisfying the commutation relation

$$[a_\Omega(k), a_\Omega^+(k')] = (2\pi)^v \, 2 \, w_k(\Omega) \, \delta^v(k' - k').$$ (8)

The upperbound of the ground state energy is given by the minimum of $\langle O_\Omega | H | O_\Omega \rangle$, where H is the Hamiltonian density of the system and $|O_\Omega\rangle$ is a normalized Gaussian wave function centered on $\varphi = \varphi_0$. We write

$$V^G(\varphi_0, \Omega(\varphi_0)) = \langle O_\Omega | H | O_\Omega \rangle$$ (9)

since H contains term like $\nabla\varphi$, φ^2, φ^4 etc. their expectation values have to be calculated to find $\langle O_\Omega | H | O_\Omega \rangle$. They can be found using the ansatz (6) and the commutation relation (8). Here we just quote the results [7]:

$$\langle O | \tfrac{1}{2}(\dot{\varphi}^2 + (\Delta\varphi)^2) | O \rangle = I_1^\beta - \frac{\Omega^2}{2} I_0,$$ (9a)

$$\langle O | \varphi^2 | O \rangle = \varphi_0^2 + I_0^\beta,$$ (9b)

$$\langle O | \varphi^4 | O \rangle = \varphi_0^4 + 6 \varphi_0^2 I_0^\beta + 3 I_0^2,$$ (9c)

$$\langle O | \varphi^6 | O \rangle = \varphi_0^6 + 15(I_0^\beta \varphi^4 + 3 I_0^\beta \varphi_0^2 + I_0^{\beta^3}),$$ (9d)

where I_n^β is given by

$$I_n^\beta = \int \frac{\mathrm{d}^v k}{(2\pi)^v \, 2 \, w_k(\Omega^2)} [w_k^2(\Omega^2)]^n.$$ (10)

(Unless otherwise stated, I_n^β is to be understood as $I_n^\beta(\Omega)$.) The integral (10) is to be calculated taking account of finite temperature after writing it in covariant form. The integral (7) satisfies the relation

$$\frac{\mathrm{d}I_n^\beta}{\mathrm{d}\Omega} = (2n-1) \, I_{n-1}^\beta(\Omega), \quad n = 0, 1, \dots .$$ (11)

Now $V_G(\varphi, \Omega)$ can be calculated using (6), and after some straightforward calculations one obtains

$$V_G(\varphi, \Omega) = I_1^\beta + \tfrac{1}{2}(m_B^2 - \Omega^2) I_0^\beta + \tfrac{1}{2} m_B^2 \varphi_0^2 - \lambda_B \varphi_0^4$$
$$+ \xi \varphi_0^6 - 6 \lambda_B \varphi_0^2 I_0^\beta - 3 \lambda_B I_0^{\beta^2}$$
$$+ 15 \xi \varphi_0^4 I_0^\beta + 45 \xi \varphi_0^2 I_0^{\beta^2} + 15 \xi I_0^{\beta^3}.$$ (12)

The optimum value $\bar{\Omega}$ of Ω is given by

$$\frac{\mathrm{d}V_G}{\mathrm{d}\Omega}\bigg|_{\Omega = \bar{\Omega}} = 0 \quad \text{and} \quad \frac{\mathrm{d}^2 V^G}{\mathrm{d}\Omega^2}\bigg|_{\Omega = \bar{\Omega}} > 0.$$ (13)

From (9) and (10) we obtain

$$\bar{\Omega}^2 = m_B^2 - 12 \lambda_B(\varphi_0^2 + I_0^\beta) + 30 \xi \varphi_0^4$$
$$+ 180 \xi \varphi_0^2 I_0^\beta + 90 \xi I_0^{\beta^2}.$$ (14)

Using (12) and (14) we can write

$$\frac{\mathrm{d}\bar{V}_G(\varphi_0)}{\mathrm{d}\varphi_0} = \varphi_0 [m_B^2 - 12 \lambda_B I_0^\beta + 90 \xi I_0^{\beta^2}$$
$$+ 4 \varphi_0^2(-\lambda_B + 15 \xi I_0^\beta) + 6 \xi \varphi_0^4],$$ (15)

where we have used the chain rule like $\dfrac{\mathrm{d}I_0^\beta}{\mathrm{d}\varphi_0} = \dfrac{\mathrm{d}I_0^\beta}{\mathrm{d}\bar{\Omega}} \dfrac{\mathrm{d}\bar{\Omega}}{\mathrm{d}\varphi_0}$ etc. Also $\bar{V}_G$ is the value of V_G when $\Omega = \bar{\Omega}$. Hence the condition for a stationary point of (15) away from the origin is $\varphi_0 = \bar{\varphi}_0$, where $\bar{\varphi}_0$ is given by

$$m_B^2 - 12 \lambda_B I_0^\beta + 90 \xi I_0^{\beta^2}$$
$$+ 4 \bar{\varphi}_0^2(-\lambda_B + 15 \xi I_0^\beta) + 6 \xi \bar{\varphi}_0^4 = 0.$$ (16)

When the minimum occurs at $\varphi = 0$ there is no symmetry breaking, but when the minimum occurs at $\varphi_0 = \bar{\varphi}_0 \neq 0$ then there is symmetry breaking and then $\bar{\Omega}$ can be written as (writing $\bar{\Omega}_s$ for the value at the stationary point)

$$\bar{\Omega}^2 = \bar{\Omega}_s^2 = 8 \bar{\varphi}_0^2(-\lambda_B + 15 \xi(I_0 + \bar{I}_0^\beta)) + 24 \xi \bar{\varphi}_0^4,$$ (17)

where I_0 and $\bar{I}_0^\beta$ are the temperature independent and temperature dependent parts of I_0^β, respectively. $\bar{I}_0^\beta$ is always finite but I_0 is a divergent integral. To make things finite, we adopt the renormaliztion

$$m_R^2 = m_B^2 - 12 \lambda_B I_0(m_R) + 90 \xi I_0^2(m_R),$$ (18a)

$$\lambda_R = \lambda_B - 15 \xi I_0(m_R).$$ (18b)

Then $\bar{\Omega}_s^2$ can be written as

$$\bar{\Omega}_s^2 = 8 \bar{\varphi}_0^2(-\lambda_R + 15 \xi \bar{I}_0^\beta + 15 \xi \Delta I_0) + 24 \xi \bar{\varphi}_0^4,$$ (19)

where $\bar{\varphi}_0$ is given by

$$m_R^2 - 12 \lambda_R(\bar{I}_0^\beta + \Delta I_0) + 90 \xi(\bar{I}_0^{\beta^2} + \Delta I_0^2 + 2 \Delta I_0 \bar{I}_0^\beta)$$
$$+ 4 \bar{\varphi}_0^2(-\lambda_R + 15 \xi \bar{I}_0^\beta + 15 \xi \Delta I_0) + 6 \xi \bar{\varphi}_0^4 = 0$$ (19a)

and

$$\Delta I_0 \equiv I_0(\bar{\Omega}_s) - I_0(m_R).$$ (20)

ΔI_0 is finite in $(1+1)$ dimension and is equal to

$$-\left(\ln \frac{\bar{\Omega}_s^2}{m_R^2}\right)\Big/ 4\pi.$$

For large T, $\bar{I}_0^\beta$ is given by [8]

$$\bar{I}_0^\beta = \frac{1}{\pi}\left[\frac{\pi T}{M} + \frac{1}{2}\ln\left(\frac{M}{4\pi T}\right) + \frac{\gamma}{2}\right] + O\left(\frac{M^2}{T^2}\right). \quad (21)$$

Beyond a certain temperature, say $T = T_c$, (19) fails to give any real value of $\bar{\Omega}_s$. For $T > T_c$, the only minimum of $\bar{V}_G(\varphi_0)$ occurs at $\varphi_0 = 0$ and the definition of $\bar{\Omega}$ changes to (writing $\bar{\Omega}_0$ for $\bar{\Omega}$ at $\varphi_0 = 0$)

$$\bar{\Omega}_0^2 = \frac{d^2 \bar{V}_G(\varphi_0, \Omega)}{d\varphi_0^2}\Bigg|_{\varphi_0 = 0}. \quad (22)$$

Then from (14) we get

$$\bar{\Omega}^2 = \bar{\Omega}_0^2 = m_R^2 - 12\lambda_R \bar{I}_0^\beta + 90\xi(\bar{I}_0^{\beta 2} + \Delta I_0^2 + 2\Delta I_0 \bar{I}_0^\beta). \quad (23)$$

In Fig. 1 we plot $\bar{\Omega}$ against temperature for $\lambda_R = 0.2$ and $\xi = 0.02$. As can be seen from Fig. 1, there is a discontinuity in the plot at $T = T_c$. For $T < T_c$, $\bar{\Omega}$ is drawn using (19). But for $T > T_c$, (19) has no solution and $\bar{\Omega}$ is drawn using (23).

3. Regularization of $\bar{V}_G(\varphi_0, \bar{\Omega})$

To see what happens in $\bar{V}^G(\varphi_0, \bar{\Omega})$ for $T < T_c$ to $T > T_c$, we first regularize $\bar{V}^G(\varphi_0, \bar{\Omega})$ using the normalization given in (18). After some straightforward calculations we obtain

$$\bar{V}^G(\varphi_0, \bar{\Omega}) = \frac{\bar{\Omega}^2 - m_R^2}{8\pi} - \frac{m_R^2 \ln x}{8\pi} + \bar{I}_1^\beta$$
$$+ \frac{1}{2}(m_R^2 - \bar{\Omega}^2)(\bar{I}_0^\beta + \Delta I_0)$$
$$+ \frac{1}{2}\varphi_0^2(m_R^2 - 12\lambda_R \bar{I}_0^\beta - 12\lambda_R \Delta I + 90\xi \Delta I_0^2$$
$$\qquad + 90\xi \bar{I}_0^{\beta 2} + 180\xi \bar{I}_0^\beta \Delta I_0)$$
$$+ \varphi_0^4(-\lambda_R + 15\xi(\bar{I}_0^\beta + \Delta I_0)) \quad (24)$$
$$+ \xi \varphi_0^6 - 3\lambda_R(\bar{I}_0^\beta + \Delta I_0)^2 + 15\xi(\bar{I}_0^\beta + \Delta I_0)^3 + D,$$

where we have used
$$I_1(\bar{\Omega}) - I_1(m_R)$$
$$= \frac{1}{2}(\bar{\Omega}^2 - m_R^2)I_0(m_R) - m_R^2\frac{(x\ln x - x + 1)}{8\pi}, \quad (25)$$

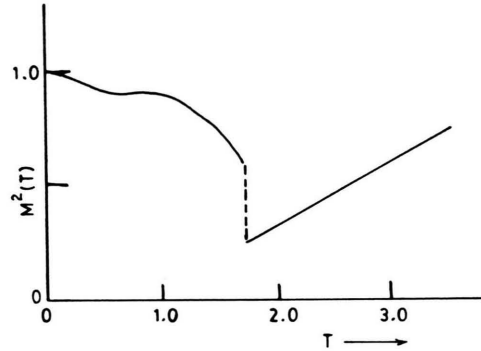$$I_0(\bar{\Omega}) - I_0(m_R) = -\frac{\ln x}{4\pi}, \quad (26)$$
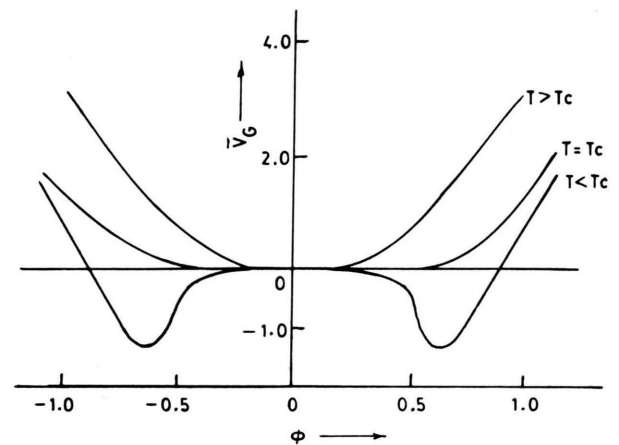

Fig. 1. Plot of mass gap.

Fig. 2. Plot of $\bar{V}_G(\varphi)$ for different temperatures.

x being $\bar{\Omega}^2/m_R^2$ and D the divergent constant in $\bar{V}_G$. Writing $\bar{V}_G^R(\varphi_0, \bar{\Omega}) = \bar{V}_G(\varphi_0, \bar{\Omega}) - D$, we see that $\bar{V}_G^R$ is now free of divergent terms (note that our normalization is different from that of Su et al. [2, 3], who have neglected the contribution of the terms given in (25) and (26)). In Fig. 2, $\bar{V}_G^R$ is plotted against φ_0 for different temperatures. It is clear that for $T < T_c$, $\bar{V}_G^R$ has a minimum at $\varphi_0 \neq 0$ but for $T \geq T_c$ the minimum shifts to $\varphi_0 = 0$. In Fig. 3, $\bar{V}_G^R$ is plotted for the particular case $\lambda_R^2 = 2\xi m_R^2$ (taking $\lambda_R = 0.19$). The structure of the two potentials is similar between $T = 0$ and $T = 0.2$, but at $T = 0.2$ the absolute minimum occurs at $\varphi_0 = 0$. The false vacua exist at $T = 0.2$ but gradually disappear as the temperature increases.

4. Discussion and Conclusions

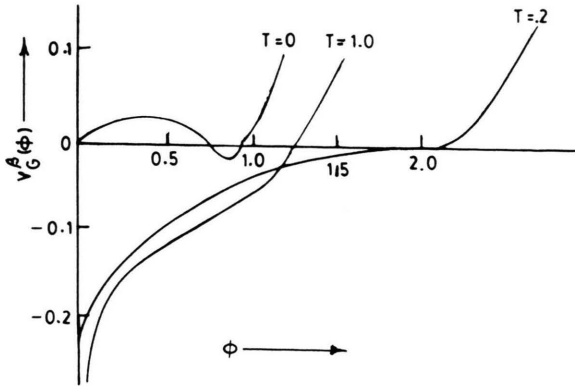In this paper we have studied soliton solutions in φ^6 field theory using the Gaussian effective potential

Fig. 3. Plot of $V_G^\beta(\varphi)$ with $\lambda = -\sqrt{2}\,\xi\,m_R$.

approach. To the best of our knowledge the mass gap equation φ^6 scalar theory was presented here for the first time using this approach. We have studied $\bar{V}_G^R$ both for the case $\lambda^2 = 2\,\xi\,m^2$ and $\lambda^2 \neq 2\,\xi\,m^2$. Our results differ from these of Su et al. [3] because they have neglected the contribution of the kinetic term which explains the absence of the $\bar{I}_1^\beta$ term in their expression. Their normalization procedure is also not rigorous. To compare our results with those of Su et al., we consider the static soliton solution. For the static soliton solution consider the equation

$$\frac{d^2\varphi_0}{dx^2} = \frac{\partial \bar{V}_G^F}{\partial \varphi_0}. \tag{27}$$

From (24) it can be seen that even in the absence of temperature the above equation is transcendental due to the presence of ΔI_0 terms in I_0 and I_1. This is in contrast with the result of Su et al., who obtained an algebraic equation for the soliton. In the ultra relativistic case, however, ΔI_0 can be neglected and (27) can be formally integrated to obtain

$$\int \frac{d\varphi}{\sqrt{2(E + U_{\text{eff}})}} = x + x_0, \tag{28}$$

where U_{eff} is obtained from $\bar{V}_G^F(\varphi_0)$. This equation, in the context of classical mechanics systems, follows from the energy conservation law. The $E = 0$ case gives a kink solution. Again, in the ultra relativistic case U_{eff} can be written as (neglecting terms independent of φ_0)

$$U_{\text{eff}}(\varphi_0) = \frac{M_T^2}{2}\,\varphi_0^2 - \lambda^T \varphi_0^4 + \xi\,\varphi_0^6, \tag{29}$$

where

$$M^T = m_R^2 - 12\,\lambda_R\,\bar{I}_0^\beta + 90\,\xi\,\bar{I}_0^\beta \tag{30a}$$

and

$$\lambda^T = \lambda_R - 15\,\xi\,\bar{I}_0^\beta. \tag{30b}$$

Hence only in the ultra relativistic case our result will agree with that of Su et al. (compare (29) and (30) with (4.1) to (4.3) of [3]). But in general $\bar{V}_G^F(\varphi_0)$ would be a transcendental function of φ_c, and the soliton solution can only be investigated by numerical analysis.

Acknowledgement

[1] R. K. Su, Pin-Zhen Bi, and Guang-Jiong Ni, J. Phys. A Math. Gen. **16**, 2445 (1983).
[2] R. K. Su and Xiang-guang Gu, J. Phys. A Math. Gen. **19**, 2891 (1986).
[3] R. K. Su and Taochen, J. Phys. A Math. Gen. **20**, 5939 (1987).
[4] J. Kuczynski, R. Manka, and J. J. Sladnowski, J. Phys. A Math. Gen. **19**, 91 (1986).
[5] R. Roychoudhury and M. Sengupta, Z. Naturforsch. **44a**, 524 (1989).
[6] I. Roditi, Phys. Letters B **294**, ■ (1986).
[7] P. M. Stevenson, Phys. Rev. D **32**, 1387 (1986).
[8] L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).